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On arc-traceable local tournaments

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Abstract

A digraph D is *arc-traceable* if for every arc xy of D, the arc xy belongs to a directed Hamiltonian path of D. A *local tournament* is an oriented graph such that the negative neighborhood as well as the positive neighborhood of every vertex induces a tournament. It is well known that every tournament contains a directed Hamiltonian path and, in 1990, Bang-Jensen showed the same for connected local tournaments. In 2006, Busch, Jacobson and Reid studied the structure of tournaments that are not arc-traceable and consequently gave various sufficient conditions for tournaments to be arc-traceable. Inspired by the article of Busch, Jacobson and Reid, we develop in this paper the structure necessary for a local tournament to be not arc-traceable. Using this structure, we give sufficient conditions for a local tournament to be arc-traceable and we present examples showing that these conditions are best possible.

Keywords: Arc-traceable; Hamiltonian path; Local tournaments

1. Terminology and introduction

For a digraph D, we denote by $V(D)$ and $E(D)$ the *vertex set* and *arc set* of D, respectively. The number $|V(D)|$ is the *order* of the digraph D. The subdigraph induced by a subset A of $V(D)$ is denoted by $D[A]$.

If $xy \in E(D)$, then y is a *positive neighbor* or *out-neighbor* of x and x is a *negative neighbor* or *in-neighbor* of y, and we also say that x *dominates* y and that y *is dominated by* x, denoted by $x \to y$. More generally, if A and B are two disjoint subdigraphs of a digraph D such that every vertex of A dominates every vertex of B, then we say that A *dominates* B and that B *is dominated by* A, denoted by $A \to B$. Furthermore, $A \rightsquigarrow B$ denotes the fact that there is no arc leading from B to A and at least one arc is leading from A to B. In this case also we say that A *weakly dominates* B. The *outset* $N^+(x)$ of a vertex x is the set of positive neighbors of x. More generally, for arbitrary subdigraphs A and B of D, the outset $N^+(A, B)$ is the set of vertices in B to which there is an arc from a vertex in A. The *insets* $N^-(x)$ and $N^-(A, B)$ are defined analogously. The numbers $d^+(x) = \left|N^+(x)\right|$ and $d^-(x) = \left|N^-(x)\right|$ are called *outdegree* and *indegree* of x, respectively. The *minimum outdegree* $\delta^+(D)$ and the *minimum indegree* $\delta^-(D)$ of D are given by $\min\left\{d^+(x)|x \in V(D)\right\}$ and $\min\left\{d^-(x)|x \in V(D)\right\}$, respectively. Furthermore, let $\delta(D)$ denote the minimum of $\delta^+(D)$ and $\delta^-(D)$.

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Throughout this paper, directed cycles and paths are simply called *cycles* and *paths*. The length of a cycle C or a path P is the number of arcs included in C or P. Let $C = x_1x_2 \dots x_kx_1$ be a cycle of length k. Then $C[x_i, x_j]$, where $1 \le i, j \le k$, denotes the subpath $x_ix_{i+1} \dots x_j$ of C with *initial vertex* x_i and *terminal vertex* x_j. Furthermore, if x is a vertex of C, then x_C^+ denotes the successor of x on C. The *predecessor* of a vertex x is defined analogously. If no confusion arises, x^+ and x^- will be used to denote x_C^+ and x_C^-. The notations for paths are defined analogously.

A digraph D is *arc-traceable* if every arc xy of D belongs to a path of order $|V(D)|$, i.e., a Hamiltonian path.

All digraphs mentioned here are finite without loops, multiple arcs and cycles of length two.

We speak of a *connected digraph* if the underlying graph is connected. A digraph D is said to be *strongly connected* or just *strong*, if for every pair x, y of vertices of D, there is a path from x to y. A *strong component* of D is a maximal induced strong subdigraph of D. A digraph D is *k-connected* if for any set S of at most $k - 1$ vertices the subdigraph $D - S$ is strong. If D is a strong digraph and S is a subset of $V(D)$ such that $D - S$ is not strong, we say that S is a *separating set*. We speak of a separating vertex s if $S = \{s\}$ is a separating set of size one. A separating set S is called *minimal separating set* (*minimum separating set*) if there exists no separating set U such that $U \subseteq S$ and $U \neq S$ ($|U| < |S|$).

An *n-tournament* is an orientation of a complete undirected graph with order n. A *local tournament* is a digraph where the inset as well as the outset of every vertex induces a tournament and an *in-tournament* is a digraph where the inset of every vertex induces a tournament.

A tournament T is called *regular* if $\delta(T) = \Delta(T)$ and *almost-regular* if $\Delta(T) - \delta(T) \le 1$.

Let D be a digraph with $V(D) = \{v_1, v_2, \dots, v_r\}$ and let $H_1, H_2, \dots, H_r$ be a collection of digraphs. Then $D[H_1, H_2, \dots, H_r]$ is the new digraph obtained from D by replacing each vertex v_i of D with H_i and adding the arcs from every vertex of H_i to every vertex of H_j if v_iv_j is an arc of D for all i and j satisfying $1 \le i \neq j \le r$. The following class of digraphs plays an important role in the study of local tournaments. A digraph on n vertices is called a *round digraph* if we can label its vertices $v_1, v_2, \dots, v_n$ such that $N^+(v_i) = \{v_{i+1}, v_{i+2}, \dots, v_{i+d^+(v_i)}\}$ and $N^-(v_i) = \{v_{i-1}, v_{i-2}, \dots, v_{i-d^-(v_i)}\}$ for every i, where the subscripts are taken modulo n. We refer to $v_1, v_2, \dots, v_n$ as a *round labeling* of D. A local tournament D is *round-decomposable* if there exists a round local tournament R on $r \ge 2$ vertices and strong local subtournaments $H_1, H_2, \dots, H_r$ of D such that $D = R[H_1, H_2, \dots, H_r]$. We call $R[H_1, H_2, \dots, H_r]$ a *round decomposition* of D.

Throughout this paper all subscripts are taken modulo the corresponding number.

In 1990, Bang-Jensen [2] defined local tournaments to be the family of oriented graphs where the inset as well as the outset of every vertex induces a tournament. In transferring the general adjacency only to vertices that have a common positive or a common negative neighbor, local tournaments form an interesting generalization of tournaments. Since then a lot of research has been done concerning local tournaments, or the more general class of *locally semicomplete digraphs*, where there might be cycles of length two. In particular, the Ph.D. Theses of Guo [9] and Huang [11] handled this subject in detail. For more information concerning different generalizations of tournaments, the reader may be referred to the survey article of Bang-Jensen and Gutin [3].

In claiming adjacency only for vertices that have a common positive neighbor, Bang-Jensen, Huang and Prisner [4] introduced a further generalization of local tournaments, the class of in-tournaments. Some problems concerning in-tournaments have been studied by Bang-Jensen, Huang and Prisner in their initial article [4].

In this paper, we develop the structure necessary for a local tournament D to contain an arc that does not belong to a Hamiltonian path. Using this structure, we give sufficient conditions for a local tournament to be arc-traceable. In addition we give examples that show that these conditions are best possible.

The first result concerning Hamiltonian paths in digraphs is due to Rédei [15] who showed that every tournament contains a Hamiltonian path.

Theorem 1.1 (*Rédei [15] 1934*). *Every tournament contains a Hamiltonian path.*

In 1990, Bang-Jensen [2] showed the same for connected local tournaments. In 1960, a first sufficient condition was given for a digraph to contain a Hamiltonian cycle (and thus, in particular a Hamiltonian path).

Theorem 1.2 (*Ghouila-Houri [8] 1960*). *If D is a digraph such that $\delta(D) \ge |V(D)|/2$, then D contains a Hamiltonian cycle.*

Moon [14] proved that every vertex in a strongly connected tournament belongs to cycles of arbitrary lengths.

Theorem 1.3 (*Moon [14] 1966*). *A tournament T is strongly connected if and only if T is vertex-pancyclic.*

Weaker results were shown by Camion [7] in 1959 and by Harary and Moser [10] in 1966.

Theorem 1.4 (*Camion [7] 1959*). *A tournament T is strongly connected if and only if T is Hamiltonian.*

Theorem 1.5 (*Harary & Moser [10] 1966*). *A tournament T is strongly connected if and only if T is pancyclic.*

In 1990, Bang-Jensen [2] extended the result of Camion to local tournaments. The result was further extended to in-tournaments by Bang-Jensen, Huang and Prisner [4] in 1993.

Theorem 1.6 (*Bang-Jensen, Huang & Prisner [4] 1993*). *An in-tournament D is strongly connected if and only if D is Hamiltonian.*

Since tournaments are known to have a Hamiltonian path and strong tournaments have Hamiltonian cycles, it is a natural question to ask under which conditions every arc of a given tournament is part of a Hamiltonian path or cycle. In fact, the condition that every arc belongs to a cycle of length k for every integer $3 \le k \le n$ was introduced as *arc-pancyclicity* and the following results were given.

Theorem 1.7 (*Alspach [1] 1967*). *Every regular tournament is arc-pancyclic.*

Theorem 1.8 (*Jakobsen [12] 1972*). *Every arc of an almost-regular n-tournament, where $n \ge 8$, is contained in a cycle of length k for every $4 \le k \le n$.*

Theorem 1.9 (*Thomassen [16] 1980*). *If T is an n-tournament with $\Delta(T) - \delta(T) \le (n-3)/5$, then every arc of T is contained in a cycle of length k for every $4 \le k \le n$.*

In contrast to arc-pancyclicity the question of arc-traceability was not addressed. In 2006, Busch, Jacobson and Reid [6] studied the structure of tournaments that are not arc-traceable and consequently gave various sufficient conditions for tournaments to be arc-traceable. They proved the following result.

Theorem 1.10 (*Busch, Jacobson & Reid [6] 2006*). *If T is a strong tournament with an arc xy that is not on a Hamiltonian path, then*

(a) *there exists a vertex z such that $T - z$ is not strong;*
(b) *$T - z$ has $p \ge 4$ strong components;*
(c) *x is in the initial strong component of $T - z$ and y is in the terminal strong component of $T - z$;*
(d) *z is dominated by the 2nd strong component of $T - z$ and z dominates the $(p-1)$th strong component of $T - z$.*

Inspired by the article of Busch, Jacobson and Reid, we develop in this paper the structure necessary for a local tournament to be not arc-traceable. In particular, we transfer Theorem 1.10 to the class of local tournaments. Using this structure, we give sufficient conditions for a local tournament to be arc-traceable and we present examples showing that these conditions are best possible.

2. Preliminary results

The results in this section will be frequently used in our proofs. For strong, but not 2-connected local tournaments, Volkmann [17] showed the following.

Theorem 2.1 (*Volkmann [17] 2000*). *Let u be a vertex of a strong local tournament such that $D - u$ is not strong. If $D_1, D_2, \ldots, D_p$ is the strong decomposition of $D - u$, then the arcs from D_i to D_{i+1} for $1 \le i \le p-1$ and the arcs in D_i for $2 \le i \le p-1$ are contained in a Hamiltonian path of D.*

The next result is a simple, but powerful observation on the interaction of a cycle and an external vertex.

Lemma 2.2 (*Bang-Jensen [2] 1990*). *Let D be a local tournament containing a cycle $C = u_1u_2 \dots u_ku_1$. If there exists a vertex $v \in V(D) - V(C)$ such that $N^+(v, C) \neq \emptyset$ (or $N^-(C, v) \neq \emptyset$), then either $v \to C$ ($C \to v$, respectively) or $u_i \to v \to u_{i+1}$ for some integer $1 \leq i \leq k$, i.e., there exists a cycle C' in D such that $V(C') = V(C) \cup \{v\}$.*

This lemma as well as Theorem 1.4 for local and in-tournaments is useful for the analysis of the structural properties of local and in-tournaments.

Theorem 2.3 (*Bang-Jensen [2] 1990*). *Let D be a strong local tournament and let S be a minimal separating set of D.*

(a) *If A and B are two distinct strong components of $D - S$, then either there is no arc between them or A dominates B or B dominates A;*
(b) *If A and B are two distinct strong components of $D - S$ such that A dominates B, then $D[A]$ and $D[B]$ are tournaments;*
(c) *The strong components of $D - S$ can be ordered in a unique way $D_1, D_2, \dots, D_p$ such that there are no arcs from D_j to D_i for $j > i$, and D_i dominates D_{i+1} for $i = 1, 2, \dots, p - 1$.*

An analogous result was given by Bang-Jensen, Huang and Prisner for in-tournaments.

Theorem 2.4 (*Bang-Jensen, Huang & Prisner [4] 1993*). *Let D be a strong in-tournament and let S be a minimal separating set of D.*

(a) *If A and B are two distinct strong components of $D - S$, either there is no arc between them or A weakly dominates B or B weakly dominates A. Furthermore, if A weakly dominates B, the set $N^-(B, A)$ dominates B.*
(b) *If A and B are two distinct strong components of $D - S$ such that A weakly dominates B, the set $N^-(b, A)$ induces a tournament for every $b \in B$.*
(c) *The strong components of $D - S$ can be ordered in a unique way $D_1, D_2, \dots, D_p$ such that there are no arcs from D_j to D_i for $j > i$, and D_i has an arc to D_{i+1} for $i = 1, 2, \dots, p - 1$.*

According to Theorems 2.3 and 2.4, the following definition was given.

Definition 2.5. Let D be a strong local or in-tournament and let S be a minimal separating set of D. Then the unique sequence $D_1, D_2, \dots, D_p$ as defined in Theorems 2.3 and 2.4, respectively, is called the *strong decomposition* of $D - S$. Furthermore, we call D_1 the *initial strong component* and D_p the *terminal strong component* of $D - S$.

The following results are immediate by Theorems 2.3 and 2.4, respectively.

Theorem 2.6 (*Bang-Jensen [2] 1990*). *Let D be a strong local tournament and let S be a minimal separating set of D. The strong decomposition of $D - S$ has the following properties.*

(a) *If $x_i \to x_j$ for $x_i \in V(D_i)$ and $x_j \in V(D_j)$ with $1 \leq i \neq j \leq p$, then $D_k \to D_l$ for every $i \leq k < l \leq j$.*
(b) *The digraph $D - S$ has a Hamiltonian path.*
(c) *For every $s \in S$ we have $d^+(s, D_1) > 0$ and $d^-(s, D_p) > 0$.*

Theorem 2.7 (*Bang-Jensen, Huang & Prisner [4] 1993*). *Let D be a strong in-tournament and let S be a minimal separating set of D. The strong decomposition of $D - S$ has the following properties.*

(a) *If $x_i \to x_k$ for $x_i \in V(D_i)$ and $x_k \in V(D_k)$ with $1 \leq i \neq k \leq p$, then $x_i \to D_j$ for every $i + 1 \leq j \leq k$.*
(b) *The digraph $D - S$ has a Hamiltonian path.*
(c) *For every $s \in S$ we have $d^+(s, D_1) > 0$ and $d^-(s, D_p) > 0$.*

The next result summarizes some useful observations regarding the order of a digraph.

Lemma 2.8. *Let D be a digraph of order n without cycles of length two and let $r \geq 1$ be an integer.*

(a) *If $d^-(x) + d^+(y) \geq r$ for every arc xy of D, then $n \geq r + 1$;*
(b) *If $\delta^+(D) \geq r$ or $\delta^-(D) \geq r$, then $n \geq 2r + 1$;*
(c) *If $\delta^+(D) \geq r$ and there exists a vertex $x \in V(D)$ such that $d^+(x) \geq r + 1$ or if $\delta^-(D) \geq r$ and there exists a vertex $x \in V(D)$ such that $d^-(x) \geq r + 1$, then $n \geq 2r + 2$.*

3. Sufficient criteria for arc-traceability

In this section we give various criteria for a strong local tournament to be arc-traceable.

3.1. Sufficient criteria for arc-traceability in terms of (local) connectivity

We begin with local tournaments that are not strongly connected.

Observation 3.1. *Let D be a connected, but not strongly connected local tournament with the strong decomposition $D_1, D_2, \ldots, D_p$, where $p \geq 2$. Then D is arc-traceable if and only if each of the strong components is arc-traceable and there exists no arc leading from D_i to D_j for $j \geq i + 2$.*

As a result of this observation, we can now focus on strongly connected local tournaments. For in-tournaments the following result is obvious by Theorem 1.6.

Observation 3.2. *If D is a strong in-tournament and u is not a separating vertex of D, then every arc incident with u belongs to a Hamiltonian path of D.*

As immediate consequences we obtain the following corollaries.

Corollary 3.3. *If D is a 2-connected in-tournament, then D is arc-traceable.*

Corollary 3.4 (*Busch, Jacobson & Reid [6] 2006*). *If T is a strong tournament and u is not a separating vertex of T, then every arc incident with u belongs to a Hamiltonian path of T.*

Corollary 3.5 (*Busch, Jacobson & Reid [6] 2006*). *If T is a 2-connected tournament, then T is arc-traceable.*

Using Theorem 2.4, we can show the following proposition.

Theorem 3.6. *If D is a strong in-tournament with the property that there exist at least two internally vertex disjoint paths from y to x for every arc xy, then D is arc-traceable.*

Proof. Let x be an arbitrary vertex of D. By Observation 3.2 it suffices to show that x is not a separating vertex of D. So assume that $D - x$ is not strong and let $D_1, D_2, \ldots, D_p$ be the strong decomposition of $D - x$, where $p \geq 2$. Let uv be an arc of D such that $u \in V(D_1)$ and $v \in V(D_2)$. Then all paths from v to u include the vertex x, a contradiction to our assumption. □

In particular we derive the following result.

Corollary 3.7 (*Busch [5] 2005*). *If T is a tournament with the property that there exist at least two internally vertex disjoint paths from y to x for every arc xy, then T is arc-traceable.*

Busch, Jacobson and Reid [6] showed that in tournaments this property applies locally as well.

Theorem 3.8 (*Busch, Jacobson & Reid [6] 2006*). *Let T be a strong tournament and let $xy \in E(T)$ be an arc of T such that there exist two internally vertex disjoint paths from y to x in T. Then there exists a Hamiltonian path through xy in T.*

However, the following example shows that Theorem 3.8 is no longer valid for local tournaments.

Example 3.9. Let T be an arbitrary tournament and let T' be a tournament on at least two vertices. We define the local tournament D by the vertex set

$$V(D) = V(T) \cup V(T') \cup \{x, y\}$$

and the edge set

$$E(D) = E(T) \cup E(T') \cup \{xy\} \cup \{xv \mid v \in V(T)\} \cup \{vy \mid v \in V(T)\} \cup \{wx \mid w \in V(T')\} \cup \{yw \mid w \in V(T')\}.$$

Then there exist two internally vertex disjoint paths from y to x in D, but there is no Hamiltonian path through xy in D.

Note that every local tournament D as defined in Example 3.9 has the following two properties. Firstly, the vertices x and y are the only separating vertices of D and, secondly, there is no arc between the internal vertices of every longest path P from y to x and $D - P$. The next results show that these properties are indeed necessary for an arc xy to be not traceable.

Theorem 3.10. *Let D be a strong local tournament with at least three separating vertices and let xy be an arc of D such that there exist at least two (internally) vertex disjoint paths from y to x in D. Then there exists a Hamiltonian path of D that includes the arc xy.*

Proof. Let $z \notin \{x, y\}$ be a separating vertex of D. Since there are at least two (internally) vertex disjoint paths from y to x in D, the vertices x and y are in the same strong component of $D - z$. Let $D_1, D_2, \ldots, D_p$ be the strong decomposition of $D - z$, where $p \geq 2$. According to Theorem 2.1, every arc in D_i for $2 \leq i \leq p - 1$ belongs to a Hamiltonian path of D. Hence we may assume that $xy \in E(D_i)$ for an index $i \in \{1, p\}$. By symmetry we may assume, without loss of generality, that $i = 1$.

Case 1: Suppose that there exists only one (internally) vertex disjoint path from y to x in D_1. Then every pair P_1, P_2 of (internally) vertex disjoint paths from y to x in D has the property that $z \in V(P_1)$ and $V(P_2) \subseteq V(D_1)$ or that $z \in V(P_2)$ and $V(P_1) \subseteq V(D_1)$. We may assume, without loss of generality, that the latter assumption holds. Assume now that we have chosen a pair P_1, P_2 of (internally) vertex disjoint path from y to x in D under the following conditions:

A. the successor of y on P_2 is not in D_1,
B. under condition A: $|V(P_1)| + |V(P_2)|$ is maximal.

It follows by condition **B** that $V(D) - V(D_1) \subseteq V(P_2)$. Let $R = V(D) - (V(P_1) \cup V(P_2))$ be the set of the remaining vertices. Since R induces a tournament in D, there exists a Hamiltonian path Q of $D[R]$ (cf. Theorem 1.1). Therefore D contains the Hamiltonian path

$$QP_2[y^+, x]P_1[y, x^-]$$

through xy.

Case 2: Suppose that there exist two (internally) vertex disjoint paths from y to x in D_1. Let $P_1 = yu_1u_2 \ldots u_s x$ and $P_2 = yw_1w_2 \ldots w_t x$ be two such paths such that

$$|V(P_1)| + |V(P_2)| \geq |V(P_1')| + |V(P_2')|$$

for any pair P_1', P_2' of (internally) vertex disjoint paths from y to x in D_1. Let $U = \{u_1, u_2, \ldots, u_s\}$, $W = \{w_1, w_2, \ldots, w_t\}$ and let $R = V(D_1) - (V(P_1) \cup V(P_2))$ be the set of the remaining vertices in D_1. If $R = \emptyset$, the path

$$P_1[u_1, x]P_2[y, w_t]$$

is a Hamiltonian path of D_1 through xy and thus, xy is also on a Hamiltonian path of D. So assume that $R \neq \emptyset$ and let $Q = q_1q_2 \ldots q_r$ be a Hamiltonian path of $D[R]$. If $q_r \to u_1$ or $q_r \to w_1$, the arc xy is on the Hamiltonian path

$$QP_1[u_1, x]P_2[y, w_t] \quad \text{or} \quad QP_2[w_1, x]P_1[y, u_s]$$

of D_1 and thus, xy is also on a Hamiltonian path of D. So $\{u_1, w_1\} \to q_r$. Let $i = \min\{j \mid N^-(q_j, U \cup W) \neq \emptyset\}$ be the minimal index such that q_i has a negative neighbor in $U \cup W$ and let $j_1 = \max\{j \mid u_j \to q_i\}$ and $j_2 = \max\{j \mid w_j \to q_i\}$ be the maximal indices such that $\{u_{j_1}, w_{j_2}\} \to q_i$. If $j_1 < s$, the path

$$P_1[u_1, u_{j_1}]q_i P_1[u_{j_1+1}, u_s]$$

together with P_2 yields a contradiction to the choice of P_1 and P_2 and if $j_2 < t$, the path

$$P_2[w_1, w_{j_2}]q_i P_2[w_{j_2+1}, w_t]$$

together with P_1 yields a contradiction to the choice of P_1 and P_2. So assume that $j_1 = s$ and $j_2 = t$. It follows that

$$Q[q_1, q_{i-1}]P_1[u_1, x]P_2[y, w_t]Q[q_i, q_r]$$

is a Hamiltonian path of D_1 through xy and thus, xy is also on a Hamiltonian path of D. □

Theorem 3.11. *Let D be a strong local tournament and let xy be an arc of D such that there exist at least two (internally) vertex disjoint paths from y to x in D. If there is a longest path P from y to x in D such that D has an arc between an internal vertex of P and $D - P$, then there exists a Hamiltonian path of D that includes the arc xy.*

Proof. Suppose that the arc xy is not traceable. Then both x and y are separating vertices of D by Observation 3.2. By Theorem 3.10 we may assume, that $D - z$ is strong for every vertex $z \notin \{x, y\}$. Let $D_1, D_2, \ldots, D_p$ be the strong decomposition of $D - x$, where $p \geq 2$. Since xy is not traceable, we conclude that $y \notin V(D_1)$. Note that if y belongs to D_i, all paths from y to x are subdigraphs of $D[V(D_i) \cup V(D_{i+1}) \cup \cdots \cup V(D_p)]$. Let P be a path from y to x in D that fulfills the assumption of this theorem.

If $2 \leq i \leq p - 1$, then P contains all vertices of $D_i, D_{i+1}, \ldots, D_p$. It follows that either there is an arc between a component D_r and a component D_s, where $1 \leq r < i < s \leq p$, or $|V(D_i)| \geq 3$ and there is an arc between a component D_r and $D_i - \{y\}$, where $r < i$. Both possibilities contradict the assumption that y is a separating vertex of D.

So assume that $i = p$. Let P_1 and P_2 be two (internally) vertex disjoint paths from y to x such that $|V(P_1)| + |V(P_2)| \geq |V(P_1')| + |V(P_2')|$ for every pair P_1', P_2' of (internally) vertex disjoint paths from y to x. Let $P_1 = yu_1u_2 \ldots u_sx$, $P_2 = yw_1w_2 \ldots w_tx$ and let $R = V(D_p) - (V(P_1) \cup V(P_2))$ be the set of the remaining vertices of D_p. If $R = \emptyset$, the path

$$D_1D_2 \ldots D_{p-1}P_1[u_1, x]P_2[y, w_t]$$

contradicts the assumption that xy is not traceable. So assume that $R \neq \emptyset$ and let $A_1, A_2, \ldots, A_q$ be the strong decomposition of $D[R]$, where $q \geq 1$. If there is an arc u_sv from u_s to A_1, let Q be a Hamiltonian path of $D[R]$ that begins in v. Then

$$D_1D_2 \ldots D_{p-1}P_2[w_1, x]P_2[y, u_s]Q$$

is a Hamiltonian path of D through xy, again a contradiction. Hence $A_1 \to u_s$. Let $y = u_0$. If there is a vertex $v \in V(A_1)$ that has an in-neighbor on $P_1[y, u_s]$, let $i = \max\{j \mid u_j \to v\}$ be the maximal index with $u_i \to v$. But then

$$P_1[y, u_i]vP_1[u_{i+1}, x]$$

and P_2 are two (internally) vertex disjoint paths from y to x that contradict the choice of P_1 and P_2. Therefore $A_1 \to P_1[y, u_s]$. Analogously we can show that $A_1 \to P_2[y, w_t]$. So all in all we obtain $A_1 \to (D_p - A_1)$, a contradiction to the assumption that D_p is strong. This final contradiction completes the proof of the theorem. □

In combining Theorems 3.10 and 3.11 we derive Theorem 3.8 as an immediate corollary.

3.2. Sufficient criteria for arc-traceability in terms of cycle lengths

In this subsection we show that the property that an arc of a strong local tournament D belongs to a cycle of length more than $(|V(D)| + 1)/2$ is sufficient for this arc to be traceable under an additional assumption. We begin with a preparatory result.

Lemma 3.12. *Let D be a strong local tournament and let xy be an arc of D. If P is a longest path from y to x in D, then P can be extended to a Hamiltonian cycle of D.*

Proof. Let P be a longest path from y to x and let

$$C = u_1u_2 \ldots u_sv_0v_1 \ldots v_ru_1,$$

where $s \geq 1$ and $r \geq 0$, be a longest cycle that includes P, i.e. $u_1 = y$, $u_s = x$ and $P = u_1u_2 \ldots u_s$. Now assume that $V(C) \neq V(D)$ and let $v \notin V(C)$ be a vertex that does not belong to C. If v has both an out- and an in-neighbor on C, it can be inserted in C by Lemma 2.2, a contradiction to either the choice of P or C. It follows that the vertex set $V(D) - V(C)$ can be partitioned in $X^+ \cup X^- \cup \tilde{X}$ such that $X^- \to C \to X^+$ and the vertices of $\tilde{X}$ have neither out- nor in-neighbors on C. Since D is strong, we conclude that $X^+ \neq \emptyset$ and $X^- \neq \emptyset$. Now let $w \in X^+$ be a vertex that dominates C and let $Q = w_0 w_1 \ldots w_t$ be a shortest path from C to $w = w_t$, where $t \geq 2$. Let w_0^+ and w_0^- be the successor and predecessor of w_0 on C. If $w_0 \in V(P) - \{x\}$, the path

$$P[y, w_0^-] Q P[w_0^+, w]$$

is a longer path from y to w than P, a contradiction. So assume that $w_0 \in V(C) - V(P)$ or $w_0 = w$. In replacing the arc $w_0 w_0^+$ by the path Q we can construct a cycle that includes P and is longer than C, the final contradiction. □

Using Lemma 3.12 for tournaments, Busch, Jacobson and Reid [6] showed the following theorem.

Theorem 3.13 (*Busch, Jacobson & Reid [6] 2006*). *Let T be a strong n-tournament and let xy be an arc of T that is on some cycle of length more than $\frac{n+1}{2}$. Then xy is on a Hamiltonian path of T.*

This theorem is no longer valid for local tournaments.

Example 3.14. Let D be the local tournament on $n \geq 4$ vertices that consists of a cycle $C = v_1 v_2 \ldots v_n v_1$ and the arc $v_1 v_3$. Then $v_1 v_3$ belongs to a cycle of length $n - 1 > (n+1)/2$, but not to a Hamiltonian path of D.

In order to transfer Theorem 3.13 to the class of local tournaments we tighten the assumption as one can see in the next result.

Theorem 3.15. *Let D be a strong local tournament and let xy be an arc of D that belongs to a longest cycle C of length $l > \frac{n+1}{2}$. If there exists an arc between $V(C) - \{x, y\}$ and $V(D) - V(C)$, then the arc xy is on a Hamiltonian path of D.*

Proof. Let $C = xyu_1u_2 \ldots u_k x$ be a longest cycle containing the arc xy. Using Lemma 3.12, we can extend the path $C - xy$ to a Hamiltonian cycle C' of D. Let

$$C' = yu_1u_2 \ldots u_k x w_m w_{m-1} \ldots w_1 y,$$

where $k + 2 > \frac{n+1}{2}$, $m \geq 1$ and $n = m + k + 2$. Since C has length $k + 2 > \frac{n+1}{2}$, it follows that $k \geq m$. Let $U = \{u_1, u_2, \ldots, u_k\}$ and $W = \{w_1, w_2, \ldots, w_m\}$.

Case 1: Suppose that there exists an arc leading from W to U. Let

$$j = \min\{r \mid N^+(w_r, U) \neq \emptyset\}$$

be the smallest integer such that the vertex w_j has at least one positive neighbor in U and let

$$i = \max\{s \mid w_j \to u_i\}$$

be the greatest integer such that w_j dominates u_i.

If $j = 1$, the path

$$w_m w_{m-1} \ldots w_1 u_i u_{i+1} \ldots u_k x y u_1 u_2 \ldots u_{i-1}$$

is a Hamiltonian path through xy.

So assume that $j \geq 2$. Note that if $u_r \to w_s \to u_{r+1}$ for two indices $1 \leq r \leq k-1$ and $1 \leq s \leq m$, the path

$$yu_1u_2 \ldots u_r w_s u_{r+1} u_{r+2} \ldots u_k x$$

is a longer path from y to x than P, a contradiction. Therefore, using the local tournament property of D, we conclude that $w_j \to u_r$ for every index $1 \leq r \leq i$. Since $w_j \to w_{j-1}$ and D is a local tournament, it follows that w_{j-1} and u_r are adjacent for every index $1 \leq r \leq i$. Due to the choice of j we have $u_r \to w_{j-1}$ for every index $1 \leq r \leq i$.

If $i \geq 2$, the path

$$w_m w_{m-1} \ldots w_j u_i u_{i+1} \ldots u_k x y u_1 u_2 \ldots u_{i-1} w_{j-1} w_{j-2} \ldots w_1$$

is a Hamiltonian path through xy.

So assume that $i = 1$. Then $k \geq m \geq 2$, since $m \geq j \geq 2$. In addition note that we have already shown that $u_1 \to w_{j-1}$. Due to the choice of P this implies that $u_2 \to w_{j-1}$ and, since $i = 1$ was chosen maximal, it follows that $u_2 \to w_j$. Now let

$$p = \max\{q \geq 0 \mid u_{1+q} \to w_{j-1+q}, u_{2+q} \to w_{j-1+q}, u_{2+q} \to w_{j+q}\}$$

be the greatest integer such that u_{1+p} dominates w_{j-1+p} and u_{2+p} dominates w_{j-1+p} and w_{j+p}.

If $p = m - j$, the path

$$u_{3+p}u_{4+p} \dots u_k x y u_1 u_2 \dots u_{2+p} w_m w_{m-1} \dots w_1$$

is a Hamiltonian path through xy.

So assume that $p < m - j$. Note that $2 + p < k$, since $k \geq m$. Due to the choice of P we obtain $u_{3+p} \to w_{j+p}$ and thus, since p was chosen maximal, we conclude that $w_{j+p+1} \to u_{3+p}$. But now

$$w_m w_{m-1} \dots w_{j+p+1} u_{3+p} u_{4+p} \dots u_k x y u_1 u_2 \dots u_{2+p} w_{j+p} w_{j-1+p} \dots w_1$$

is a Hamiltonian path through xy.

Case 2: Suppose that $U \rightsquigarrow W$. Let $u_r \to w_s$, where $1 \leq r \leq k$ and $1 \leq s \leq m$, be an arc from U to W. Using the local tournament property of D and the assumption that $U \rightsquigarrow W$, it follows that $u_r \to w_m$. But then

$$u_{r+1} u_{r+2} \dots u_k x y u_1 u_2 \dots u_r w_m w_{m-1} \dots w_1$$

is a Hamiltonian path through xy which completes the proof of this theorem. □

The following results are immediate consequences of Theorem 3.15.

Corollary 3.16 (*Busch, Jacobson & Reid [6] 2006*). *Let T be a strong tournament and let $xy \in E(T)$ be an arc of T that belongs to a cycle of length $l > \frac{n+1}{2}$. Then the arc xy is on a Hamiltonian path of T.*

Corollary 3.17. *If D is a strong local tournament and every arc xy of D is on a longest cycle C of length $l > \frac{n+1}{2}$ such that there exists an arc between $V(C) - \{x, y\}$ and $V(D) - V(C)$, then D is arc-traceable.*

Corollary 3.18 (*Busch, Jacobson & Reid [6] 2006*). *If T is a strong tournament and every arc of T is on some cycle of length $l > \frac{n+1}{2}$, then T is arc-traceable.*

4. Structure of local tournaments with a non-traceable arc

In this section we prove some important necessary conditions for local tournaments to have a non-traceable arc. These conditions are used in Section 5 to obtain various sufficient conditions for arc-traceability.

Theorem 4.1. *Let D be a strong local tournament and let xy be an arc of D that does not belong to a Hamiltonian path. Then D satisfies one of the following conditions.*

(a) (i) *D is a round-decomposable local tournament with the round decomposition $R[H_1, H_2, \dots, H_r]$, where $r \geq 4$, such that $V(H_i) = \{x\}$ and $V(H_j) = \{y\}$, where $|i - j| \geq 2$;*
  (ii) *$D - \{x, y\}$ is not connected.*

(b) (i) *there exists a vertex z such that $D - z$ is not strong;*
  (ii) *$D - z$ has $p \geq 4$ strong components;*
  (iii) *x is in the initial strong component of $D - z$ and y is in the terminal strong component of $D - z$;*
  (iv) *z has no out-neighbor in the 2nd strong component of $D - z$ and z has no in-neighbor in the $(p-1)$th strong component of $D - z$.*

Proof. Let xy be an arc of D that is not traceable. According to Observation 3.2, both x and y are separating vertices of D.

Case 1: Suppose that D has a separating vertex $v \notin \{x, y\}$. Then, in view of Theorem 3.10, there is at most a single vertex disjoint path leading from y to x in D. Now, by Menger's Theorem [13], there is a y–x-separating set of size one. Let $z \notin \{x, y\}$ be a separating vertex of D such that there exists no path leading from y to x in $D - z$ and let $D_1, D_2, \dots, D_p$ be the strong composition of $D - z$, where $p \geq 2$. According to Theorem 2.1, the vertices y and x are not in consecutive strong components of $D - z$ which immediately implies that $p \geq 3$. Furthermore, in view of Theorem 2.3, the vertex z has at least one positive neighbor in D_1 and at least one negative neighbor in D_p. Let $x \in V(D_\alpha)$ and $y \in V(D_\beta)$.

If $\alpha > 1$, we conclude that $V(D_\alpha) = \{x\}$, since x is a separating vertex of D. Analogously we obtain $V(D_\beta) = \{y\}$ if $\beta < p$.

If there is an arc leading from D_i to D_j such that $i < \alpha < j$ or $i < \beta < j$, the vertex x or y, respectively, is not a separating vertex of D, a contradiction. If $\alpha > 1$ and there is an arc leading from z to D_j such that $\alpha < j$, the digraph $D - x$ is strong, a contradiction. If $\beta < p$ and there is an arc leading from D_i to z such that $i < \beta$, the digraph $D - y$ is strong, again a contradiction. So assume the contrary.

Subcase 1.1: Suppose that $\alpha > 1$ and $\beta < p$. Then the round decomposition $R[H_1, H_2, \ldots, H_r]$ of D can be constructed as follows.

> [**Construction of the round decomposition of** D.]
> Let $H_i = D_i$ for $i = 1, 2, \ldots, p$, $H_{p+1} = \{s\}$, where s is the considered separating vertex of D, and $r = p + 1$. Let R be the local tournament with vertex set $\{H_1, H_2, \ldots, H_r\}$ and arc set $\{H_i H_j \mid N^+(H_i, H_j) \neq \emptyset\}$. Then $R[H_1, H_2, \ldots, H_r]$ is the desired round decomposition of D.

Subcase 1.2: Suppose that $\alpha > 1$ and $\beta = p$ or $\alpha = 1$ and $\beta < p$. Without loss of generality, we may assume the latter. By the observations above we conclude that $z \to D_1$. Since x is a separating vertex of D, it follows that $V(D_1) = \{x\}$. In addition, if there is an arc from z to D_j for $j < \beta$, we obtain $z \to D_2$. In this case

$$xyD_{\beta+1}D_{\beta+2}\ldots D_p z D_2 D_3 \ldots D_{\beta-1}$$

is a Hamiltonian path of D through xy, a contradiction. So there is no arc between z and D_j for $1 < j < \beta$ and the round decomposition $R[H_1, H_2, \ldots, H_r]$ of D can be constructed as above.

Subcase 1.3: Suppose that $\alpha = 1$ and $\beta = p$. Since x and y are both separating vertices of D, we conclude that $z \not\to D_1$ if $|V(D_1)| \geq 3$ and $D_p \not\to z$ if $|V(D_p)| \geq 3$. In addition, z has no positive neighbor in D_2 and no negative neighbor in D_{p-1}. If $p \geq 4$, the local tournament D has the desired structure (b). So assume that $p = 3$. Then z has neither an out- nor an in-neighbor in D_2. It follows that $D_3 \to z \to D_1$. By the observations above we conclude that $|V(D_1)| = |V(D_3)| = 1$ and the round decomposition $R[H_1, H_2, H_3, H_4]$ of D can be constructed as above.

Case 2: Suppose that $D - v$ is strong for every vertex $v \notin \{x, y\}$. Let $D_1, D_2, \ldots, D_p$ be the strong decomposition of $D - x$, where $p \geq 2$, and let $y \in V(D_\beta)$. Since xy is not traceable, we obtain $\beta > 1$.

Subcase 2.1: Suppose that $\beta < p$. If $|V(D_\beta)| \geq 3$, the path

$$D_1 D_2 \ldots D_{\beta-1} C_\beta[y^+, y^-] D_{\beta+1} D_{\beta+2} \ldots D_p xy$$

is a Hamiltonian path through xy, a contradiction. So assume that $|V(D_\beta)| = 1$. Since y is a separating vertex of D, we conclude that D has no arc leading from D_i to D_j for $i < \beta < j$. If x has a negative neighbor in D_i for an index $i < \beta - 1$, the vertex x has negative neighbors both in D_i and D_p. It follows that $D_{\beta-1} \to D_{\beta+1}$, a contradiction to the assumption that y is a separating vertex of D. So $x \to D_i$ for every index $i \leq \beta$. We can analogously show that x has no negative neighbors in D_j for every index $j > \beta$. This particularly implies that $D_p \to x$. So the round decomposition $R[H_1, H_2, \ldots, H_r]$ of D can be constructed as above.

Subcase 2.2: Suppose that $\beta = p$. Then $|V(D_p)| \geq 3$, since x has a negative neighbor in D_p. Now, by Menger's Theorem [13] there are two (internally) vertex disjoint path from y to x. Observe that all internal vertices of these paths are in D_p and that $D_{p-1} \to D_p$. Hence xy is traceable by Theorem 3.11, the final contradiction. □

We make the following observations.

Remark 4.2. Every strong round-decomposable local tournament that satisfies condition (a) of Theorem 4.1 is not a tournament. Furthermore, every strong tournament that satisfies condition (b) of Theorem 4.1 is not round-decomposable. In other words: if D is a strong round-decomposable tournament, then D is arc-traceable.

As an immediate consequence of the above theorem and remark we obtain Theorem 1.10. A deeper analysis of the structure of strong local tournaments that have a non-traceable arc and are not round-decomposable yields the following result.

Theorem 4.3. *Let D be a strong local tournament that is not round-decomposable and let xy be a non-traceable arc of D. Let $z \notin \{x, y\}$ be a separating vertex of D and let $D_1, D_2, \ldots, D_p$ be the strong decomposition of $D - z$, where $p \geq 4$.*

(a) *If* $\delta(D) \geq 2$*, then the components* D_1 *and* D_p *both have at least* $2\delta(D)+1$ *vertices;*
(b) *If* $D_1 - x$ *induces a strong tournament in* D*, then* x *is the only out-neighbor of* z *in* D_1*;*
(c) *If* x *is a separating vertex of* D_1*, then all out-neighbors of* z *in* $D_1 - x$ *are in the terminal strong component of* $D_1 - x$*;*
(d) *If* $D_p - y$ *induces a strong tournament in* D*, then* y *is the only in-neighbor of* z *in* D_p*;*
(e) *If* y *is a separating vertex of* D_p*, then all in-neighbors of* z *in* $D_p - y$ *are in the initial strong component of* $D_p - y$*.*

Proof. Assume that $D_1 - x$ is strong. If z has an out-neighbor w in $D_1 - x$, let C_p be a Hamiltonian cycle of D_p, let v be an in-neighbor of z in D_p and let C_1 be a Hamiltonian cycle of $D_1 - x$. Then the path

$$xC_p[y, v]zC_1[w, w^-]D_2D_3 \ldots D_{p-1}C_p[v^+, y^-]$$

shows that xy is traceable, a contradiction. Hence z has no out-neighbors in $D_1 - x$. So (b) is valid and by symmetry (d) is also valid.

To prove (c) assume that x is a separating vertex of D_1. Let $A_1, A_2, \ldots, A_q$ be the strong decomposition of $D_1 - x$, where $q \geq 2$. If z has an out-neighbor w in A_i, where $i < q$, let C_p be a Hamiltonian cycle of D_p, let v be an in-neighbor of z in D_p, let C_i be a Hamiltonian cycle of A_i and let Q be a Hamiltonian path of $D_1 - (V(A_i) \cup \{x\})$ that begins in A_1 and ends in an in-neighbor u of x in A_q. Then the path

$$QxC_p[y, v]zC_i[w, w^-]D_2D_3 \ldots D_{p-1}C_p[v^+, y^-]$$

shows that xy is traceable, again a contradiction. Hence z has no out-neighbors in A_i for every $i < q$. So (c) is proved and by symmetry we derive the validity of (e).

Now let $\delta(D) \geq 2$. Then $|V(D_1)| \geq 3$. If $D_1 - x$ is strong, we deduce from (b) and Lemma 2.8(c) that $|V(D_1) - \{x\}| \geq 2\delta(D)$ and hence $|V(D_1)| \geq 2\delta(D) + 1$. If x is a separating vertex of D_1, we obtain $|V(A_1)| \geq 2\delta(D) - 1$ by (c) and Lemma 2.8(b). It follows that $|V(D_1)| \geq |V(A_1)| + |V(A_2)| + |\{x\}| \geq 2\delta(D) + 1$. By symmetry we obtain $|V(D_p)| \geq 2\delta(D) + 1$ which completes the proof of (a) and of this theorem. □

5. Applications of the structural results

In this section we shall apply the structural results of Section 4 to present several sufficient conditions for the arc-traceability of strongly connected local tournaments. Firstly we consider strongly connected, round-decomposable local tournaments. Using Lemma 2.8(a) and (b), we can show the following results.

Theorem 5.1. *Let* D *be a strong round-decomposable local tournament that is not arc-traceable and let* k *be the number of separating vertices of* D*. If* $\delta(D) \geq 2$ *and* $d^-(u) + d^+(v) \geq s$ *for every arc* uv *in* D*, then* $ks \leq |V(D)|$*.*

Proof. If $s = 1$, the proposition is immediate. So assume that $s \geq 2$. Let $R[H_1, H_2, \ldots, H_r]$ be the round decomposition of D, where $r \geq 4$. Note that an arbitrary Hamiltonian cycle of D has no consecutive separating vertices, since $\delta(D) \geq 2$. Let xy be a non-traceable arc of D. Since D is round-decomposable, it has the structure as described in Theorem 4.1(a). So let, without loss of generality, $V(H_1) = \{x\}$ and $V(H_i) = \{y\}$, where $3 \leq i \leq r - 1$. Then H_2 contains a vertex u and H_{i-1} contains a vertex v such that $d^-(u) \leq (|V(H_2)| + 1)/2$, $d^+(v) \leq (|V(H_{i-1})| + 1)/2$ and $u \to v$. It follows that

$$\begin{aligned} d^-(u) + d^+(v) &\leq \frac{|V(H_2)| + 1}{2} + \frac{|V(H_{i-1})| + 1}{2} \\ &= \frac{|V(H_2)| + |V(H_{i-1})|}{2} + 1 \\ &\leq \frac{|V(D)| - k - (k-2)(s-1)}{2} + 1. \end{aligned}$$

The last inequality is true because of Lemma 2.8(a). Since $s \leq d^-(u) + d^+(v)$, we conclude from the last inequality that $ks \leq |V(D)|$. □

Similarly we can show the following result with the help of Lemma 2.8(b).

Theorem 5.2. *Let D be a strong round-decomposable local tournament that is not arc-traceable and let k be the number of separating vertices of D. If $\delta(D) \geq 2$, then $2k\delta(D) \leq |V(D)|$.*

Proof. By Observation 3.2, we assume that $k \geq 2$. Let $R[H_1, H_2, \ldots, H_r]$ be the round decomposition of D, where $r \geq 4$. Note that an arbitrary Hamiltonian cycle of D has no consecutive separating vertices, since $\delta(D) \geq 2$. Since D is round-decomposable, it has the structure as described in Theorem 4.1(a). So let, without loss of generality, x and y be two separating vertices of D such that $V(H_1) = \{x\}$ and $V(H_i) = \{y\}$, where $3 \leq i \leq r-1$. Then H_2 contains a vertex u such that $\delta(D) \leq d^-(u) \leq (|V(H_2)| + 1)/2$. It follows that

$$\delta(D) \leq \frac{|V(H_2)| + 1}{2} \leq \frac{|V(D)| - (k-1) - (k-1)(2\delta(D) - 1)}{2}.$$

The last inequality is true because of Lemma 2.8(b). We conclude from the last inequality that $2k\delta(D) \leq |V(D)|$. □

As immediate corollaries we state the following sufficient conditions for a strong round-decomposable local tournament to be arc-traceable.

Corollary 5.3. *Let D be a strong round-decomposable local tournament on n vertices with k separating vertices. If $\delta(D) \geq 2$ and $d^-(x) + d^+(y) \geq \frac{n+1}{k}$ for every arc xy of D, then D is arc-traceable.*

Proof. If D is not arc-traceable, it follows by Theorem 5.1 that

$$n \geq \frac{n+1}{k} k = n + 1,$$

a contradiction. □

Corollary 5.4. *Let D be a strong round-decomposable local tournament on n vertices with k separating vertices. If $\delta(D) \geq 2$ and $\delta(D) \geq \frac{n+1}{2k}$, then D is arc-traceable.*

Proof. If D is not arc-traceable, it follows by Theorem 5.2 that

$$n \geq \frac{n+1}{2k} 2k = n + 1,$$

a contradiction. □

We now consider local tournament that are not round-decomposable.

Theorem 5.5. *Let D be a strong local tournament on n vertices that is not round-decomposable with $\delta(D) \geq 2$. If $d^-(x) + d^+(y) \geq \frac{n}{2} - 2$ for every arc xy of D, then D is arc-traceable.*

Proof. We show that if D is not traceable, it has an arc xy with $d^-(x)+d^+(y) < \frac{n}{2}-2$. So suppose that xy is an arc of D that is not traceable. Then D has the structure as described in Theorem 4.1(b). Note that $n \geq |V(D_1)|+|V(D_p)|+3$ and that $D_1 \to D_p$, since $x \to y$. Let u be a vertex in D_1 with minimal indegree and let v be a vertex in D_p with minimal outdegree. Since $D_1 \to D_p$, the arc uv exists in D. As $\delta(D) \geq 2$ we obtain $|V(D_1)| \geq 3$ and $|V(D_p)| \geq 3$.

If D_1 is regular or almost-regular, the vertex x is not a separating vertex of D_1. It follows by Theorem 4.3(b) that $(D_1 - x) \rightsquigarrow z$. Analogously, if D_p is regular or almost-regular, it follows by Theorem 4.3(d) that $z \rightsquigarrow (D_p - y)$. Hence $d^-(u) \leq (|V(D_1)| - 1)/2$ and $d^+(v) \leq (|V(D_p)| - 1)/2$.

If D_1 is neither regular nor almost-regular, it is immediate that $d^-(u) \leq (|V(D_1)| - 1)/2$. Analogously we see that $d^+(v) \leq (|V(D_p)| - 1)/2$ if D_p is neither regular nor almost-regular.

So all in all we conclude that $d^-(u) \leq (|V(D_1)| - 1)/2$ and $d^+(v) \leq (|V(D_p)| - 1)/2$. It follows that

$$\begin{aligned} d^-(u) + d^+(v) &\leq \frac{|V(D_1)| - 1}{2} + \frac{|V(D_p)| - 1}{2} \\ &= \frac{|V(D_1)| + |V(D_p)| - 2}{2} \\ &\leq \frac{n-5}{2} \\ &< \frac{n}{2} - 2 \end{aligned}$$

the desired contradiction that completes the proof of this theorem. □

Since every strong, round-decomposable tournament is arc-traceable by Remark 4.2, the next results follow directly from Theorem 5.5.

Corollary 5.6 (*Busch, Jacobson & Reid [6] 2006*). *Let T be a strong tournament on n vertices with $\delta(T) \geq 2$. If $d^-(x) + d^+(y) \geq \frac{n}{2} - 2$ for every arc xy of T, then T is arc-traceable.*

Corollary 5.7. *Let D be a strong local tournament on n vertices that is not round-decomposable. If $\delta(D) \geq \frac{n}{4} - 1 > 1$, then D is arc-traceable.*

Corollary 5.8 (*Busch, Jacobson & Reid [6] 2006*). *Let T be a strong tournament on n vertices. If $\delta(T) \geq \frac{n}{4} - 1 > 1$, then T is arc-traceable.*

The next example shows that the bounds presented in Corollaries 5.3 and 5.4 are best possible.

Example 5.9. Let $T_1, T_2, \ldots, T_k$ be σ-regular tournaments, where $\sigma \geq 1$, and let $u_1, u_2, \ldots, u_k$ be k additional vertices. We define D as the local tournament with vertex set $V(D) = \bigcup_{i=1}^{k} V(T_i) \cup \bigcup_{i=1}^{k} \{u_i\}$ and arc set

$$E(D) = \bigcup_{i=1}^{k} E(T_i) \cup \bigcup_{i=1}^{k} \{u_i v \mid v \in V(T_i)\} \cup \bigcup_{i=1}^{k} \{v u_{i+1} \mid v \in V(T_i)\} \cup \bigcup_{i=1}^{k} \{u_i u_{i+1}\}.$$

Then D is a strong round-decomposable local tournament such that $u_1, u_2, \ldots, u_k$ are the separating vertices of D and $\delta(D) \geq \sigma + 1$. Note that $d^-(x) + d^+(y) = 2\sigma + 2 = \frac{n}{k}$ for all vertices $x, y \in V(T_i)$, that $d^-(u_i) + d^+(y) = d^-(x) + d^+(u_{i+1}) = 3\sigma + 3 \geq \frac{n}{k}$ for every vertex $x \in V(T_i)$ and $y \in V(T_{i+1})$ and that $d^-(u_i) + d^+(u_{i+1}) = 4\sigma + 4 \geq \frac{n}{k}$ for every index $1 \leq i \leq k$. But D is not arc-traceable, since none of the arcs $u_i u_{i+1}$ belongs to a Hamiltonian path of D.

Now the following question arises: if D is a strong local tournament with minimal degree less than $n(D)/4$, which length has a longest path through any given arc of D? The next result answers the question.

Theorem 5.10. *Let D be a strong local tournament on n vertices with minimum degree $1 < \delta(D) < \frac{n}{4}$.*

(a) *If D is round-decomposable, then every arc of D is on a path of length at least $2\delta(D)$;*
(b) *If D is not round-decomposable, then every arc of D is on a path of length at least $\lceil \frac{n+1}{2} \rceil + 2\delta(D)$.*

Proof. If D is arc-traceable, every arc of D is on a Hamiltonian path and the result is immediate. So assume that D has a non-traceable arc xy. Then both x and y are separating vertices of D by Observation 3.2 and D has the structure required by Theorem 4.1. We consider two cases.

Case 1: Suppose that D is round-decomposable. Let $R[H_1, H_2, \ldots, H_r]$ be the round decomposition of D, where $r \geq 4$, such that, without loss of generality, $V(H_1) = \{x\}$ and $V(H_i) = \{y\}$ for an index $3 \leq i \leq r - 1$. Let $D' = D[V(H_{i+1}) \cup V(H_{i+2}) \cup \cdots \cup V(H_{p-1})]$ be the local tournament that is induced by $H_{i+1}, H_{i+2}, \ldots, H_{p-1}$. Note that $N^+(D') - V(D') \subseteq \{x\}$ and $N^-(D') - V(D') \subseteq \{y\}$. It follows by Lemma 2.8(b) that $|V(D')| \geq 2\delta(D) - 1$. Therefore the arc xy is on a path with at least $2\delta(D) + 1$ vertices and hence has length at least $2\delta(D)$.

Case 2: Suppose that D is not round-decomposable.

We shall show first that z has an out-neighbor outside of D_1. If $|V(D_1)| = 1$, the proposition is immediate, since $\delta(D) \geq 2$. So assume that $|V(D_1)| \geq 3$. If $z \to D_1$, the vertex x is not a separating vertex of D, a contradiction. Therefore z has an in-neighbor v in D_1. Since $v \to D_i$ for $i > 1$, it follows by the local tournament property of D that z is adjacent to every vertex $w \in V(D_i)$ for $i > 1$. Recall that z has no in-neighbor in D_{p-1}. Hence we obtain $z \to D_{p-1}$. Analogously we can show that z has an in-neighbor outside of D_p.

Let $r = \max\{i < p \mid N^-(z, D_i) \neq \emptyset\}$ be the greatest index less than p such that z has an in-neighbor in D_i. Dually, let $s = \min\{1 < j \mid N^+(z, D_j) \neq \emptyset\}$ be the smallest index greater than 1 such that z has an out-neighbor in D_j. Observe that $r \geq 2$ and $s \leq r + 1$. Let $v_1 \in V(D_r)$ and $v_2 \in V(D_p)$ be two vertices that dominate z and let $w_1 \in V(D_1)$ and $w_2 \in V(D_s)$ be two vertices that are dominated by z. Furthermore, let C_i be a Hamiltonian cycle of D_i for every index $1 \leq i \leq p$. Then

$$P = C_1[x^+, x]C_p[y, v_2]zC_s[w_2, w_2^-]D_{s+1}D_{s+2}\ldots D_{p-1}C_p[v_2^+, y^-]$$

and

$$Q = C_1[x^+, w_1^-]D_2D_3 \dots D_{r-1}C_r[v_1^+, v_1]zC_1[w_1, x]C_p[y, y^-]$$

are two paths through xy of order $1+|V(D_1)|+\sum_{i=s}^{p}|V(D_i)|$ and $1+|V(D_p)|+\sum_{i=1}^{r}|V(D_i)|$, respectively. Hence

$$\begin{aligned} |V(P)|+|V(Q)| &\geq |V(D_1)|+|V(D_p)|+2+\sum_{i=1}^{p}|V(D_i)| \\ &= n+1+|V(D_1)|+|V(D_p)| \\ &\geq n+3+4\delta(D) \end{aligned}$$

by Theorem 4.3(a). It follows that either P or Q contains at least $\lceil\frac{n+3}{2}\rceil+2\delta(D)$ vertices and thus has length at least $\lceil\frac{n+1}{2}\rceil+2\delta(D)$. $\square$

Since every strong, round-decomposable tournament is arc-traceable by Remark 4.2, the next results follow directly from Theorem 5.5.

Corollary 5.11 (*Busch, Jacobson & Reid [6] 2006*). *Let T be a strong local tournament on n vertices with minimum degree $1 < \delta(T) < \frac{n}{4}$. Then every arc of T is on a path of length at least $\lceil\frac{n+1}{2}\rceil+2\delta(D)$.*

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